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THE HEXAGONAL HONEYCOMB CONJECTURE

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Abstract. It is conjectured that the planar hexagonal honeycomb provides the least-perimeter way to enclose and separate infinitely many regions of unit area. Various natural formulations of the question are not known to be equivalent. We prove existence for two formulations. Many questions remain open.

1. Introduction

In 1994, D. Weaire and R. Phelan improved on Lord Kelvin's candidate for the least-area way to partition space into regions of unit volume ([WP], [AKS]; see [M2, 13.13], [P1]). Contrary to popular belief (see [W], p. 85), even the planar question remains open:

1.1. **Hexagonal Honeycomb Conjecture.** *The planar hexagonal honeycomb of Figure 1 provides the least-perimeter way to enclose and separate infinitely many regions of unit area.*

L. Fejes Tóth ([FT1, III.9, p. 84], see also [FT2, Sect. 26, Cor., p. 183]), together with a truncation argument, provides a proof for polygonal regions. In particular, among lattices of congruent tiles (easily shown to be convex), the conjecture holds, although among general tilings by congruent tiles, it remains open (cf. [CFG], Problem C15). Fejes Tóth also observes that the bees' 3-dimensional honeycomb is not optimal [FT3].

There are many versions of the conjecture. Do you allow regions of several components? empty space? Do you minimize a limiting perimeter-to-area ratio? or do you simply minimize perimeter inside large balls (for fixed areas and boundary conditions)? These versions are not known to be equivalent.

We prove existence of a solution for two versions of the problem. Both require each region to be connected. Theorem 2.1 allows empty space and minimizes a limiting perimeter-to-area ratio. Theorem 3.4 does not allow empty space and minimizes perimeter inside large balls.

In $\mathbf{R}^3$ and above, disconnected regions occur as limits of connected regions, and none of our results apply.

Finally in Section 4 we give our favorite formulation of the Hexagonal Honeycomb Conjecture, including uniqueness among a large class of competitors, allowing empty space and disconnected regions.

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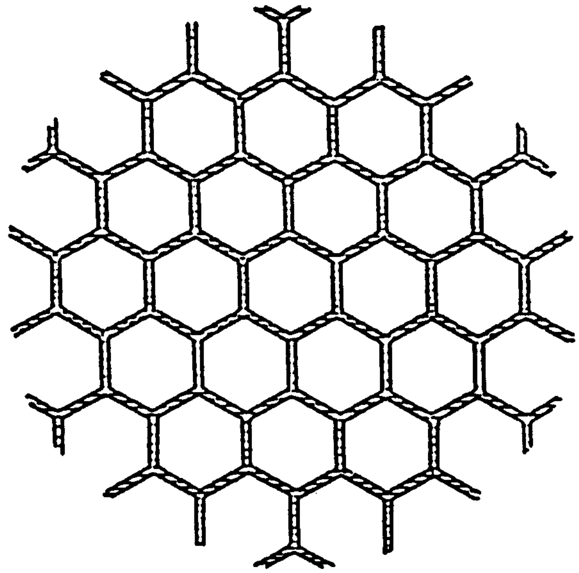

FIGURE 1. The planar hexagonal honeycomb is conjectured to be the least perimeter way to enclose and separate infinitely many regions of unit area [FT2, p. 207]. [From L. Fejes Tóth, *Regular Figures*, International Series of Monographs on Pure and Applied Mathematics, vol. 48, Macmillan, New York, 1994, p. 207. Used with permission.]

1.2. **Proofs.** The proof of Theorem 2.1 shows that the smallest limiting perimeter-to-area ratio actually can be attained by a string of finite clusters going off to infinity, each obtained by truncating an infinite cluster of small limiting perimeter-to-area ratio at a carefully chosen radius. Subsequent remarks discuss variations and open questions.

Theorem 3.4 provides existence and regularity for finite immersed graphs with faces of unit area, for which no compact admissible alterations reduce perimeter. The proof considers a sequence of clusters which are minimizing in larger and larger balls. One has control over perimeter in increasingly large balls, but one needs control over perimeter in balls of fixed size. Such control comes from the simple measure-theoretic Lemma 3.1, which says that if a measure on $\mathbf{R}^n$ satisfies $\mu(\mathbf{B}(0,r)) \leq \rho\mathcal{L}^n(\mathbf{B}(0,r))$ for fixed ρ for large r, then for some point a, $\mu(\mathbf{B}(a,r)) \leq 6^n\rho\mathcal{L}^n(\mathbf{B}(a,r))$ for all r. Regularity follows from known regularity for finite clusters.

1.3. **Finite clusters** (see [M2, Chapt. 13]). The existence and almost-everywhere regularity of least-perimeter enclosures of k regions of prescribed volume is known in general dimensions ([A], [M3], [T]). The only proven minimizers are the sphere in general dimensions, the standard double bubble in $\mathbf{R}^2$ [Fo], and the equal-volumes standard double bubble in $\mathbf{R}^3$ ([HHS], [HS]; see [P2], [M1]). Only in the plane does the theory admit requiring each region to be connected [M3, Cor. 3.3]. In general it remains conjectural that in any minimizer each region is connected and there is no empty space trapped inside.

1.4. **Acknowledgments.** This work was inspired by the AMS special session on Soap Bubble Geometry at the Burlington Mathfest, 1995, and by discussions with John Sullivan. It was supported by the National Science Foundation.

2. Existence of infinite clusters minimizing the limiting perimeter-to-area ratio

Theorem 2.1 proves the existence of a nice planar cluster of infinitely many connected regions of unit area minimizing a limiting perimeter-to-area ratio. Our solution consists of a string of finite clusters stretching off to infinity, surrounded by lots of empty space. This theorem reduces the Hexagonal Honeycomb Conjecture to favorable comparison with finite n-clusters, which we show holds for $n \leq 398$ (Corollary 2.5).

Existence remains conjectural without allowing empty space (Remark 2.9), or allowing disconnected regions (Remark 2.6), or worse allowing higher dimensions (Remark 2.7).

2.1. Theorem. *Let $\mathcal{C}_n$ $(1 \leq n \leq \infty)$ denote the collection of planar clusters C formed by piecewise smooth planar curves Γ such that $\mathbf{R}^3 - \Gamma$ has precisely n bounded connected components, all of unit area. Let*

$$\rho_n = \inf_{C \in \mathcal{C}_n} \frac{\text{perimeter } (C)}{n}, \qquad 1 \leq n < \infty,$$

$$\rho_\infty = \inf_{C \in \mathcal{C}_\infty} \limsup_{r \to \infty} \frac{\text{perimeter } (C \cap \mathbf{B}(0,r))}{\text{area } (C \cap \mathbf{B}(0,r))}.$$

(*By* [M3, Cor. 3.3], *there is a limiting immersed minimizer M_n with perimeter $n\rho_n$, in which edges might bump up against each other.*)

Then $\rho_\infty = \lim_{n\to\infty} \rho_n$, and ρ_∞ is attained by a $C \in \mathcal{C}_\infty$ consisting of infinitely many $C_{n_i} \in \mathcal{C}_{n_i}$ (and similarly by an infinite disjoint union of M_{n_i}). Each $\rho_n > (1 + 1/\sqrt{n})\sqrt{\pi}$ and $1.77 \approx \sqrt{\pi} \leq \rho_\infty \leq \sqrt[4]{12} \approx 1.86$.

Proof. Consider a sequence $C_k \in \mathcal{C}_\infty$ with limiting perimeter-to-area ratio

$$\rho(C_k) \equiv \limsup_{r\to\infty} \frac{L_k(r)}{A_k(r)} < \rho_\infty + 1/k,$$

where

$$L_k(r) = \text{perimeter } ((C_k \cap \mathbf{B}(0,r)),$$

$$A_k(r) = \text{area } ((C_k \cap \mathbf{B}(0,r)).$$

Choose a_k such that $L_k(a_k) > 0$, $A_k(a_k) > k$, and whenever $r \geq a_k$, then

$$\frac{L_k(r)}{A_k(r)} < \rho_\infty + 1/k.$$

For almost all r, $L_k'(r)$ exists and bounds the number of points where Γ_k meets the circle $\mathbf{S}(0,r)$, by the co-area formula [M2, 3.8]. We can choose such a $b_k \geq a_k$ such that $L_k'(b_k) < A_k(b_k)/2k$. Otherwise, for almost all $r \geq a_k$,

$$L_k' \geq A_k/2k > \frac{L_k}{2k(\rho_\infty + 1/k)}.$$

Since $L_k(r)$ is increasing, for large r,

$$L_k(r) \geq L_k(a_k) e^{r/2k(\rho_\infty + 1/k)} > (\rho_\infty + 1/k)\pi r^2,$$

which implies $A_k(r) > \pi r^2$, a contradiction.

Let C'_k consist of the regions of C_k entirely within $\mathbf{B}(0, b_k)$ and their boundaries. Since the circle $\mathbf{S}(0, b_k)$ meets Γ_k in at most $L'_k(b_k)$ points and $L'_k(b_k) < A_k(b_k)/2k$, at most $A_k(b_k)/2k$ regions which partially meet $\mathbf{B}(0, b_k)$ are discarded, and

$$\begin{aligned}\frac{\text{perimeter } (C'_k)}{\text{area } (C'_k)} &\leq \frac{L_k(b_k)}{A_k(b_k) - A_k(b_k)/2k} \leq \frac{L_k(b_k)}{A_k(b_k)} \frac{1}{1 - 1/2k} \\ &\leq \frac{\rho_\infty + 1/k}{1 - 1/2k} \leq \rho_\infty + \varepsilon_k\end{aligned}$$

with $\varepsilon_k \downarrow 0$. In particular, $\liminf_{n\to\infty} \rho_n \leq \rho_\infty$.

Next we show that $\lim_{n\to\infty} \rho_n = \liminf_{n\to\infty} \rho_n$. Choose ρ_N near $\liminf \rho_n$ and consider ρ_{kn+r} for k large and $0 \leq r \leq N-1$. Now k copies of an efficient cluster in $\mathcal{C}_N$, together with r circles of area 1, constitute an admissible cluster in $\mathcal{C}_{kN+r}$ and thus show that ρ_{kN+r} is close to $\liminf \rho_n$.

Finally we prove simultaneously that $\rho_\infty \leq \lim \rho_n \equiv \rho_*$ and that ρ_∞ is attained as asserted, by showing how to use any sequence C'_k of finite clusters with perimeter-to-area ratio less than $\rho_* + \varepsilon_k$ with $\varepsilon_k \downarrow 0$ to construct an infinite cluster in $\mathcal{C}_\infty$ with ratio equal to ρ_*.

Starting at the origin and moving toward infinity, string out lots of copies of C'_2 followed by lots of copies of C'_3 followed by lots of copies of C'_4, etc. Take enough copies of C'_2 so that beyond them the perimeter-to-area ratio stays below $\rho_* + \varepsilon_2$ as you move through the first copy of C'_3, and hence as you move through all copies of C'_3. Take enough copies of C'_3 so that the perimeter-to-area ratio gets below $\rho_* + \varepsilon_3$ and stays below $\rho_* + \varepsilon_3$ as you move through copies of C'_4. Continue. As $r \to \infty$, the perimeter-to-area ratio approaches ρ_* as desired. Therefore $\rho_\infty = \rho_*$ and ρ_∞ is attained as asserted. Of course in the proof each C'_k may be replaced by a minimizer M_k.

To show $\rho_n > (1 + 1/\sqrt{n})\sqrt{\pi}$, let $C \in \mathcal{C}_n$ be a cluster of n regions R_i. Since every portion of the perimeter of C is on the boundary of two regions or on the boundary of one region and the boundary of $\bigcup R_i$,

$$\operatorname{Per}(C) = \frac{1}{2}\left(\sum \operatorname{Per}(R_i) + \operatorname{Per}\left(\bigcup R_i\right)\right).$$

By the isoperimetric inequality,

$$\operatorname{Per}(C) \leq \frac{n}{2}(2\sqrt{\pi}) + \frac{1}{2}\left(2\sqrt{n\pi}\right),$$

with equality only if all the regions are circular, which is impossible. Therefore $\rho_n > (1 + 1/\sqrt{n})\sqrt{\pi}$.

It follows that $\rho_\infty = \lim \rho_n \geq \sqrt{\pi}$. Since the hexagonal honeycomb has limiting perimeter-to-area ratio of $\sqrt[4]{12}$, $\rho_\infty \leq \sqrt[4]{12}$.

2.2. *Remark on empty chambers.* The same result and proof hold if the clusters are allowed additional connected components ("empty chambers") of arbitrary area which does not count in the perimeter-to-area ratio. (There are still just finitely many combinatorial types of clusters in $\mathcal{C}_n$.) It is conjectured that even if allowed, such empty chambers would not occur in a minimizer M_n.

2.3. *Remark on choice of origin.* The minimizing C of Theorem 2.1 will attain ρ_∞ for every choice of origin if in the proof the C'_k go off to infinity rapidly, so that from another choice of origin the latter ones are still encountered in the same order, and if you take enough copies of C'_k so that the perimeter-to-area ratio stays below $\rho_\infty + \varepsilon_k$ as you move through the first copy of C'_{k+1} even ignoring the favorable effect of the additional area, so that the way you come at C'_{k+1} does not matter.

2.4. *Remark on periodic clusters.* Theorem 2.1 implies that the optimal limiting perimeter-to-area ratio ρ_∞ can be approached by periodic clusters, for example by centering finite clusters on any sufficiently large lattice.

2.5. Corollary. *For any planar cluster C, let*

$$\rho(C) = \limsup_{r\to\infty} \frac{\text{perimeter } (C \cap \mathbf{B}(0,r))}{\text{area } (C \cap \mathbf{B}(0,r))}.$$

To prove that the hexagonal honeycomb H minimizes ρ among partitions of almost all of $\mathbf{R}^2$ into connected regions of unit area, it suffices to prove that any finite cluster C of n connected (or if you prefer not necessarily connected) regions of unit area has $\rho(C) \geq \rho(H) = \sqrt[4]{12}$. This holds for $n \leq 398$.

Proof. Everything follows immediately from Theorem 2.1 and the observation that the lower bound on $\rho(C)$ satisfies $(1 + 1/\sqrt{n})\sqrt{\pi} \geq \sqrt[4]{12}$ for $n \leq 398$.

2.6. *Remark on disconnected regions.* I believe that the hexagonal honeycomb is minimizing in the larger category of clusters with not necessarily connected regions (see Conjecture 4.1), but even the existence of such a minimizer remains open. The proof of Theorem 2.1 does not easily generalize because there is no bound on the number of regions affected by truncation.

2.7. *Remark on higher dimensions.* The current proof of Theorem 2.1 does not apply to clusters in higher dimensions minimizing a limiting area-to-volume ratio, because a bound on the size of the slices by spheres of the interfaces does not yield a bound on the number of regions meeting the sphere.

2.8. *Remark on unequal areas.* Theorem 2.1 and its proof generalize to the case of finitely many prescribed areas $a_1, \ldots, a_k$ and prescribed probabilities $p_1 + \cdots + p_k = 1$ such that as $r \to \infty$ the fraction of regions within $\mathbf{B}(0,r)$ of area a_i approaches p_i. For approximately equal areas, the minimizer probably consists essentially of regular hexagons, as G. Fejes Tóth [FT] has shown can be accomplished by partitions of the plane. Figures 2ab show two other candidate minimizers for two equally likely prescribed areas ($k = 2, a_1 > a_2, p_1 = p_2 = 1/2$), which do better than regular hexagons. Figure 2a does better approximately for $.117 < a_2/a_1 < .206$, while Figure 2b does better approximately for $a_2/a_1 < .039$. Figure 2c never does better.

2.9. *Remark on partitions.* It remains unproved whether there is a partition of almost all of $\mathbf{R}^2$, even in the category of connected regions, which minimizes the limiting perimeter-to-area ratio. A cutting and pasting argument seems too expensive.

2.10. *Remark on regularity.* Since compact alterations of an infinite cluster do not affect the limiting perimeter-to-area ratio, no general regularity can be expected.

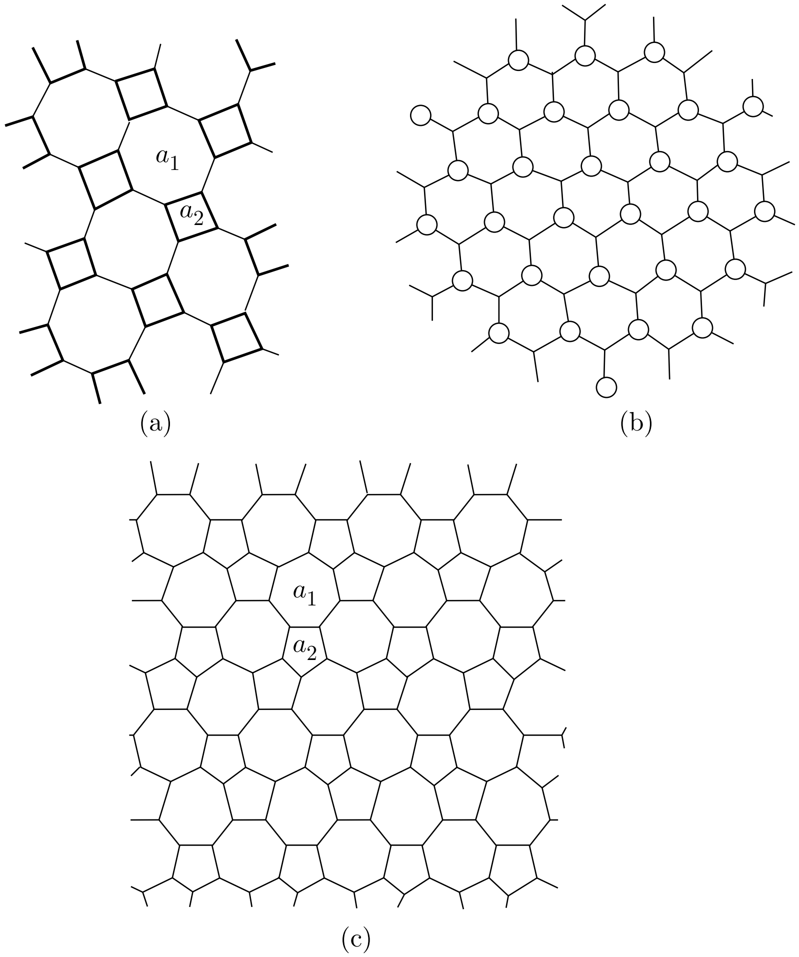

FIGURE 2. Candidate minimizers for two prescribed areas $a_1 > a_2$. Some figures based on [GS]. [From Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, Freeman, New York, 1987. Used with permission.]

3. Existence and regularity of infinite clusters minimizing perimeter inside large balls for fixed boundary

Theorem 3.4 provides nice limits G of embedded infinite planar graphs with faces of unit area, such that for balls $B, G \cap B$ is perimeter-minimizing for fixed areas and boundary. I do not know how to prove that such a G also minimizes the perimeter-to-area ratio of Section 2. It is conjectured that the Hexagonal Honeycomb H minimizes both perimeter (for fixed boundary) and perimeter-to-area ratio. One could ask further whether H is the unique perimeter minimizer.

The proof requires two lemmas. Lemma 3.1 paves the way for compactness arguments by implying uniform bounds on perimeter inside balls of any given radius.

3.1. Lemma. *Let μ be a measure on $\mathbf{R}^n$ and let R, ρ be positive numbers such that for all $r \geq 2R$,*

$$\frac{\mu(\mathbf{B}(0,r))}{\mathcal{L}^n(\mathbf{B}(0,r))} \leq \rho.$$

Then there exists a point $a \in \mathbf{B}(0,R)$ such that for all r,

$$\frac{\mu(\mathbf{B}(a,r))}{\mathcal{L}^n(\mathbf{B}(a,r))} \leq 6^n \rho.$$

Proof. Otherwise about each point $a \in \mathbf{B}(0,R)$ there is a ball $\mathbf{B}(a,r)$ with $\frac{\mu(\mathbf{B}(a,r))}{\mathcal{L}^n(\mathbf{B}(a,r))} > 6^n\rho$. Moreover $r \leq R$, since otherwise

$$\frac{\mu(\mathbf{B}(0,r+R))}{\mathcal{L}^n(\mathbf{B}(0,r+R))} > \frac{6^n \rho r^n}{(r+R)^n} \geq 3^n\rho > \rho,$$

a contradiction. Successively choose $a_i \in \mathbf{B}(0,R)$, $0 < r_i \leq R$, such that

$$\frac{\mu(\mathbf{B}(a_i,r_i))}{\mathcal{L}^n(\mathbf{B}(a_i,r_i))} > 6^n\rho,$$

the $\mathbf{B}(a_i,r_i)$ are disjoint and r_i is more than half as large as possible. We claim that $\{\mathbf{B}(a_i,3r_i)\}$ covers $\mathbf{B}(0,R)$. Suppose $b \in \mathbf{B}(0,R) - \bigcup \mathbf{B}(a_i,3r_i)$. Since there exists $0 < s \leq R$ with

$$\frac{\mu(\mathbf{B}(b,s))}{\mathcal{L}^n(\mathbf{B}(b,s))} > 6^n\rho$$

and $\mathbf{B}(b,s) \notin \{\mathbf{B}(a_i,r_i)\}$, $\mathbf{B}(b,s)$ must intersect some $\mathbf{B}(a_i,r_i)$ with $r_i > s/2$, so $b \in \mathbf{B}(a_i,3r_i)$, establishing the claim. Now

$$\begin{aligned}
\mu(\mathbf{B}(0,2R)) &\geq \sum \mu(\mathbf{B}(a_i,r_i)) \\
&> 6^n\rho \sum \mathcal{L}^n(\mathbf{B}(a_i,r_i)) \\
&= 2^n\rho \sum \mathcal{L}^n(\mathbf{B}(a_i,3r_i)) \\
&\geq 2^n\rho\mathcal{L}^n(\mathbf{B}(0,R)) \\
&= \rho\mathcal{L}^n(\mathbf{B}(0,2R)),
\end{aligned}$$

the desired contradiction.

3.2. **Definitions of spaces $\mathcal{G}_0$ and $\mathcal{G}_1$ of embedded and immersed infinite graphs.** Let $\mathcal{G}_0$ denote the set of locally finite, embedded, locally rectifiable graphs G in $\mathbf{R}^2$ with closed faces F^α ($\alpha = 1,2,3,\dots$) (including their edges) of unit area and vertices of degree at least three. Let $\mathcal{G}_1$ denote the set of locally finite limits of sequences $G_i \in \mathcal{G}_0$ such that on each sequence of increasing balls $\mathbf{B}(0,R_j)$ with $R_j \to \infty$, the G_i are eventually isotopic and converge uniformly to G.

Remarks. In the limit, edges can overlap, shrink to length 0, extend to infinity, or disappear wholly or partly at infinity. A limiting "face" has area less than or equal to 1, with equality unless it extends to infinity, in which case it need not be connected.

Lemma 3.3 isolates an easy bound on the number of faces (of unit area) which intersect a ball.

3.3. Lemma. *Let G be a graph in $\mathcal{G}_0$. Then the number $N(r)$ of faces F^α of G which intersect $\mathbf{B}(0,r)$ satisfies*

$$N(r) \leq \pi(R+I)^2 + \textit{perimeter } (G \cap B(0,r+1)) + 1.$$

Proof. Since each face of any $G \in \mathcal{G}_0$ has unit area, at least $N(r) - \pi(r+1)^2$ of the faces extend outside $\mathbf{B}(0,r+1)$. We may assume $N(r) - \pi(r+1)^2 > 1$, since otherwise the conclusion is immediate (here we need the $+1$ term in the conclusion). Since each such face has perimeter at least 2 inside $\mathbf{B}(0,r+1)$, and each edge bounds 2 faces,

$$\text{perimeter } (G \cap \mathbf{B}(0,r+1)) \geq N(r) - \pi(r+1)^2,$$

as desired.

Theorem 3.4 proves the existence of nice immersed planar graphs with faces of unit area minimizing perimeter inside compact sets. The hexagonal honeycomb is conjectured to be one such minimizer, although perhaps not the only one. It remains open whether a minimizer would also minimize the perimeter-to-area ratio of Section 2, even among partitions of almost all of $\mathbf{R}^2$; a minimizer might have increasingly inconvenient boundary conditions in larger and larger balls.

3.4. Theorem. *There exists an immersed graph $G \in \mathcal{G}_1$, $G = \lim G_i \in \mathcal{G}_0$, with the following length-minimizing property: for any ball B, if $G' \in \mathcal{G}_1$, $G' = \lim G'_i \in \mathcal{G}_0$, and $G'_i = G_i$ outside B, then*

$$\text{length}\, G \cap B \leq \text{length}\, G' \cap B. \tag{1}$$

G consists of disjoint or coincident circular arcs or straight line segments meeting

$$\textit{at vertices of } G \textit{ with the unit tangent vectors summing to } 0, \tag{2}$$

$$\textit{at other points where the edges remain } C^1 \ (\textit{actually } C^{1,1}). \tag{3}$$

Remark. If two or more vertices of G coincide, then (2) means that all of the remaining unit tangent there sum to 0.

Proof. Choose $\rho > 0$ and a sequence of graphs $G_i^0 \in \mathcal{G}_0$ and $R_i \uparrow \infty$ such that for all $r \geq 2R_i$,

$$\text{perimeter } (G_i^0 \cap \mathbf{B}(0,r)) \leq \rho(\pi r^2)$$

and

$$\text{perimeter } (G_i^0 \cap \mathbf{B}(0,2R_i)) \leq \text{perimeter } (G'_i \cap \mathbf{B}(0,2R_i)) + 1/i$$

for all G'_i in $\mathcal{G}_0$ which coincide with G_i^0 outside $\mathbf{B}(0,2R_i)$. For example, take any $\rho > \sqrt[4]{12}$ and let G_i^0 be the hexagonal honeycomb, altered inside $\mathbf{B}(0,R_i)$. By Lemma 3.1 there is a translation G_i of G_i^0 by a distance less than R_i such that for all $r > 0$,

$$\text{perimeter } (G_i \cap \mathbf{B}(0,r)) \leq 6^2 \rho\pi r^2. \tag{4}$$

We still have that

$$\text{perimeter } (G_i \cap \mathbf{B}(0,R_i)) \leq \text{perimeter } (G'_i \cap \mathbf{B}(0,R_i)) + 1/i \tag{5}$$

for all $G'_i \in G_0$ which coincide with G_i outside $\mathbf{B}(0,R_i)$.

Only finitely many faces of G_i meet a ball $\mathbf{B}(0,r)$ by Lemma 3.3. For each i, reindex the faces F_i^α in the order of their proximity to the origin. By Lemma 3.3 and (4), for each $r>0$ there is an integer A such that for all i,

$$F_i^\alpha \cap \mathbf{B}(0,r) \neq \varnothing \text{ only if } \alpha \leq A. \tag{6}$$

Of course, since each F_i^α has unit area, there are $r^\alpha > 0$ such that for all i,

$$F_i^\alpha \cap \mathbf{B}(0,r^\alpha) \neq \varnothing. \tag{7}$$

For fixed α, consider the sequence of embedded faces F_i^α. For $R>0$, condition (4) provides a bound on the number and length of components of $\partial F_i^\alpha \cap \mathbf{B}(0,2R)$ which intersect $\mathbf{B}(0,R)$. By an argument based on the compactness of Lipschitz maps, we may assume they converge uniformly. It follows with the help of (6) that the G_i and F_i^α converge to an immersed locally rectifiable graph G with "faces" F^α, whose interiors partition $\mathbf{R}^2 - G$, still satisfying (4), (6) (for a possibly larger values of A), and (7). The length-minimizing property (1) follows from condition (5).

To prove that G is locally finite, we now show that the number of vertices of G_i in a $\mathbf{B}(0,R)$ is bounded. By (2), only faces F_i^α with $\alpha \leq A$ meet $\mathbf{B}(0,R+1)$. By (4) and the co-area formula [M2, 3.8], there exists $R < R_i < R+1$ such that $\partial\mathbf{B}(0,R_i)$ meets G_i in $P_i \leq L_R \equiv 6^2\rho\pi(R+1)^2$ points. Let H_i be the restriction of G_i to $\mathbf{B}(0,R_i)$, together with the P_i arcs of the circle of radius R_i. By Euler's formula, the numbers of vertices, edges, and faces of H_i satisfy $v_i - e_i + f_i = 1$. Since every vertex has degree at least 3, $e_i \geq \frac{3}{2}v_i$, so that $v_i \leq 2(f_i - 1) \leq 2f_i \leq 2(A+P_i) \leq 2(A+L_R)$ because adding the P_i arcs added at most P_i faces. Therefore the (smaller) number of vertices of G_i inside $\mathbf{B}(0,R)$ is bounded by $2(A+L_R)$ and the limit G is locally finite as desired.

Now choose $0 < R_1 < R_2 < R_3 < \cdots \to \infty$ such that $\partial\mathbf{B}(0,R_j)$ intersects G in finitely many points, none of which are vertices. For each j, slightly alter G_i for large i such that $G_i \cap \mathbf{B}(0,R_j)$ converges uniformly to $G \cap \mathbf{B}(0,R_j)$. In particular, if in the limit an edge of G locally meets $\mathbf{B}(0,R_j)$ in a single point, slightly alter G_i for large i such that the corresponding edge locally meets $\mathbf{B}(0,R_j)$ in a nearby single point. If in the limit an edge of G locally crosses $\partial\mathbf{B}(0,R_j)$, slightly alter G_i for large i such that the corresponding edge locally crosses $\mathbf{B}(0,R_j)$ at a single nearby point.

Since $G \cap \mathbf{B}(0,R_j)$ is of finite type, there are only finitely many possible types for $G_i \cap \mathbf{B}(0,R_j)$ for large i, and we may assume that the $G_i \cap \mathbf{B}(0,R_j)$ are isotopic for large i.

Now the regularity (2), (3) follows as in [M3, Thm. 3.2, p. 356].

3.5. *Remark.* As in Section 2 (see Remarks 2.6, 2.7), little is known about related problems in which the regions of unit area may have more than one component or in higher dimensions, but I do not believe anything new occurs. For minimizers with disconnected regions, the regularity theory breaks down, because the numbers of regions need not be locally finite and hence Almgren's argument ([A, VI.2(3)] or [M2, 13.5]) on the possibility of area adjustments at cost

$$\Delta\,\mathrm{Per} \leq C\Delta\mathrm{Area}$$

does not apply.

4. Hexagonal Honeycomb Conjecture

Here we give our favorite formulation of the Hexagonal Honeycomb Conjecture, including uniqueness in a very general class of competitors, allowing empty space and disconnected regions. Explanatory remarks follow. The essential idea of allowing competitors to add more regions in empty space to improve the local perimeter-to-area ratio is due to John Sullivan.

4.1. Hexagonal Honeycomb Conjecture. *Let $\mathcal{C}$ be the collection of infinite planar clusters C defined by a 1-dimensional set S and a selection of infinitely many, not necessarily connected, disjoint, unit-area components R_i of $\mathbf{R}^2 - S$. For any closed ball B, define a perimeter-to-area ratio*

$$\rho(C,B) = \frac{\mathcal{H}^1(S \cap B)}{\mathcal{L}^2\left(\bigcup R_i \cap B\right)}.$$

Then the hexagonal honeycomb H is (up to translation and sets of measure 0) the unique $C \in \mathcal{C}$ which for each B minimizes $\rho(C,B)$ in competition with all $C' \in \mathcal{C}$ which agree with C outside B. Moreover, H minimizes $\liminf_{r\to\infty} \rho(C, \mathbf{B}(0,r))$ in competition with all $C' \in \mathcal{C}$.

4.2. *Remarks.* Minimizing every $\rho(C,B)$ is a stronger hypothesis than those of Theorems 2.1 or 3.4. Theorem 2.1 considered only a limiting ratio over balls of radius $r \to \infty$, for which a minimizer is not unique. Theorem 3.4 did not allow empty space as we do here. Here, to have hope of uniqueness, we allow competitors to add more regions R, so that something like a halfspace of hexagons (which probably minimizes $\liminf_{r\to\infty} \rho(C, \mathbf{B}(0,r))$, can be beaten by adding part of another layer.

I do not know how to prove existence or regularity of a minimizer (see Remarks 2.4, 3.5), or that such a minimizer minimizes $\liminf_{r\to\infty} \rho(C, \mathbf{B}(0,r)))$ (or $\limsup$) too (see paragraph preceding Theorem 3.4).

4.3. **General norms.** More generally one could replace perimeter by the integral of some norm applied to the unit tangent vector, as in crystals (see [MFG]). For example, for the l^1 norm, single-region minimizers are squares, which (unlike discs) tile the plane, and the analogous Square Honeycomb Conjecture follows immediately.

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